CONCURSUL INTERJUDEŢEAN DE MATEMATICĂ

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Clasa a XI a

1. Fie $A, B \in M_n(\mathbb{R})$ cu $A^2+B^2=(2+\sqrt{3})(BA-AB)$. Dacă $\det(A^2+B^2)\neq 0$, demonstraţi că n se divide cu 12.

Prof. Daniela Covaci, Brăila

***Soluție*:**

Fie $X, \overline{X} \in M_n(\mathbb{C}), X=A+Bi, \overline{X}=A-Bi \Rightarrow X\cdot\overline{X}=A^2+B^2+i(BA-AB)=(2+\sqrt{3}+i)(BA-AB)(2p)$

De unde $\det(X\overline{X})=\left|\det X\right|^2=(2+\sqrt{3}+i)^n\cdot\det(BA-AB).(1p)$

Observăm că $\left|\det X\right|^2\in\mathbb{R}, \det(A^2+B^2)\neq 0 \Rightarrow \det(AB-BA)\neq 0 \Rightarrow (2+\sqrt{3}+i)^n\in\mathbb{R}$. Dar pentru

$z=2+\sqrt{3}+i, \sin\frac{\pi}{12}=\frac{\sqrt{2}(\sqrt{3}-1)}{4}, \cos\frac{\pi}{12}=\frac{\sqrt{2}(\sqrt{3}+1)}{4}\Rightarrow tg\frac{\pi}{12}=\frac{\sqrt{3}-1}{\sqrt{3}+1}=2-\sqrt{3}=\frac{1}{2+\sqrt{3}}(1p)$

$\Rightarrow \arg z=\frac{\pi}{12}(2p)\Rightarrow z^n=r^n(\cos\frac{n\pi}{12}+i\sin\frac{n\pi}{12})\in\mathbb{R}\Rightarrow n$ divizibil cu 12. $(1p)$

2. Fie şirul $(x_n)_{n\in\mathbb{N}}$ cu $x_0>0, x_{n+1}=\left(a+x_n^5\right)^{\frac{1}{5}}, n\in\mathbb{N}, a>0.$ Calculaţi $\lim\limits_{n\to\infty}\frac{x_1+x_2+...+x_n}{n\sqrt[5]{n}}$.

Prof. Daniela Covaci, Brăila

***Soluție*:**

$x_{n+1}^5=a+x_n^5\Rightarrow x_{n+1}^5-x_n^5=a$ constantă $\Rightarrow\left(x_n^5\right)_n$ este progresie aritmetică de raţie a$(1p)$ de unde

$x_n^5=x_0^5+na(1p)\Rightarrow\lim\limits_{n\to\infty}\frac{x_n^5}{n}=a\Rightarrow\lim\limits_{n\to\infty}\frac{x_n}{\sqrt[5]{n}}=\sqrt[5]{a}\,(1p)$

$\lim\limits_{n\to\infty}\frac{x_1+x_2+...+x_n}{n\sqrt[5]{n}}=(1p)\lim\limits_{n\to\infty}\frac{x_n}{n\sqrt[5]{n}-(n-1)\sqrt[5]{n-1}}=\lim\limits_{n\to\infty}\frac{x_n}{\sqrt[5]{n}}\cdot\frac{\sqrt[5]{n}}{n\sqrt[5]{n}-(n-1)\sqrt[5]{n-1}}=(1p)$

$$= \sqrt[5]{a} \lim_{n\to\infty} \frac{\sqrt[5]{n}\left(\sqrt[5]{n^{24}} + \sqrt[5]{n^{18}(n-1)^6} + \ldots + \sqrt[5]{(n-1)^{24}}\right)}{n^6 - (n-1)^6} (1p)$$

$$= \sqrt[5]{a} \lim_{n\to\infty} \frac{n^5\left(1 + \sqrt[5]{\left(1-\frac{1}{n}\right)^6} + \sqrt[5]{\left(1-\frac{1}{n}\right)^{2\cdot 6}} + \ldots + \sqrt[5]{\left(1-\frac{1}{n}\right)^{24}}\right)}{n^6 - n^6 + C_6^1 n^5 - C_6^2 n^4 + \ldots} = \frac{5\sqrt[5]{a}}{6} (1p).$$

3. Fie matricea $A \in M_n(\mathbb{R})$. Să se determine $n \in \mathbb{N}^*$ dacă $\det\left({}^tA + iA\right) \in \mathbb{R}^*$.

Gheorghe Alexe, Brăila

***Soluţie*:**

Avem $\overline{d} = \overline{\det\left({}^tA + iA\right)}(1p) = \det\left({}^tA - iA\right)(1p) = \det{}^t\left({}^tA - iA\right) = \det\left(A - i\left({}^tA\right)\right)(1p) = (-i)^n \cdot$ $\det\left({}^tA - \frac{1}{i}A\right) = (1p) = (-i)^n \det\left({}^tA + iA\right)(1p) = (-i)^n d\,(1p), \overline{d} = d \Rightarrow (-i)^n = 1 \Rightarrow n = 4k.(1p)$