CONCURSUL INTERJUDEŢEAN DE MATEMATICĂ

“ PETRU MOROŞAN-TRIDENT ”

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Clasa a X a

1. Se consideră numerele complexe $z_n \in \mathbb{C}, n \in \{1,2,...,2020\}$ cu proprietatea $z_n = z_{n-1} \cdot z_n + 1$, oricare ar fi $n = \overline{2,2020}.$ Arătaţi că produsul $\prod_{k=1}^{2019} z_k$ este real şi calculaţi suma $\sum_{i=2}^{2020} \frac{1}{|z_{i-1}z_i| + |z_{i-1}| + 1}$.

Prof.Roxandra Murea, Brăila

Soluţie:

Observăm că $z_n \neq 1, z_n \neq 0, \forall n \in \mathbb{N}^*$.

$z_n = z_{n-1} \cdot z_n + 1,$ înmulţim relaţia cu z_{n+1}

$z_n \cdot z_{n+1} = z_{n-1} \cdot z_n \cdot z_{n+1} + z_{n+1} (1p) \Leftrightarrow z_{n-1} \cdot z_n z_{n+1} = z_{n+1} \cdot z_n - z_{n+1} = z_{n+1}(z_n - 1) \Leftrightarrow z_{n-1} \cdot z_n z_{n+1} = -1,$

oricare ar fi $n = 2, 2020.(1p)$ pentru că $z_{n+1} = z_n z_{n+1} + 1 \Leftrightarrow z_{n+1}(z_n - 1) = -1$

De unde $\prod_{k=1}^{2019} z_k = (-1)^{\frac{2019}{3}} = (-1)^{673} = -1 \in \mathbb{R}.(2p)$

Demonstrăm că $z_{n+3} = z_n$.

$$z_{n+1} = \frac{1}{1 - z_n} \Rightarrow z_{n+2} = \frac{1}{1 - z_{n+1}} = \frac{1}{1 - \frac{1}{1 - z_n}} = -\frac{1}{z_n} + 1 \Rightarrow z_{n+3} = \frac{1}{1 - z_{n+2}} = z_n \,(1p)$$

Atunci $\sum_{i=2}^{2020} \frac{1}{|z_{i-1}z_i| + |z_{i-1}| + 1} = 673\left(\frac{1}{|z_1z_2| + |z_1| + 1} + \frac{|z_1|)1}{|z_2z_3| + |z_2| + 1} + \frac{|z_1z_2|)1}{|z_3z_1| + |z_3| + 1}\right)(1p)$

$$= 673\left(\frac{1}{|z_1z_2| + |z_1| + 1} + \frac{|z_1|}{|z_1z_2| + |z_1| + 1} + \frac{|z_1z_2|}{|z_1z_2| + |z_1| + 1}\right) = 673.(1p)$$

2. Fie $a, b, c \in \mathbb{R}$ cu proprietatea că $5a + 7b + 5c = 99$ astfel încât $|7a - 5b| \leq 5, |5b - 7c| \leq 7$ și $|c - a| \leq 1$. Arătați că $99 \leq a^2 + b^2 + c^2 \leq 100.$

Prof. Carmen şi Viorel Botea, Brăila

Soluție:

$\left(7a-5b\right)^2+\left(5b-7c\right)^2+\left(5c-5a\right)^2\leq 25+49+25=99\left(1p\right)$

$99^2=\left(5a+7b+5c\right)^2\leq\left(5^2+7^2+5^2\right)\left(a^2+b^2+c^2\right)\left(1p\right)\Rightarrow 99\leq a^2+b^2+c^2\left(1p\right)$

$\left(7a-5b\right)^2+\left(5b-7c\right)^2+\left(5c-5a\right)^2+\left(5a+7b+5c\right)^2=99\left(a^2+b^2+c^2\right)\left(2p\right)\Rightarrow 99\left(a^2+b^2+c^2\right)\leq 99+$

$+99^2\left(1p\right)\Rightarrow a^2+b^2+c^2\leq 1+99\Rightarrow a^2+b^2+c^2\leq 100.\left(1p\right)$

3. Se consideră triunghiul ABC cu laturile AB, AC, BC de lungimi c, b, a diferite, care verifică relaţia $\lg^3 a+\lg^3 b+\lg^3 c=3\cdot\lg a\cdot\lg b\cdot\lg c$. Arătaţi că $S_{ABC}\leq\min\left\{\frac{1}{2a},\frac{1}{2b},\frac{1}{2c}\right\}$.

Prof.Roxandra Murea, Brăila

***Soluție*:**

Notăm $x=\lg a, y=\lg b, z=\lg c\Rightarrow x^3+y^3+z^3=3xyz$

Dar

$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)\Leftrightarrow\frac{1}{2}(x+y+z)[(x-y)^2+(y-z)^2+(z-x)^2]\Leftrightarrow$

$\Leftrightarrow x=y=z(F)$ sau $x+y+z=0\Rightarrow\lg a+\lg b+\lg c=0\Leftrightarrow abc=1.\left(2p\right)$

$S_{ABC}=\sqrt{p(p-a)(p-b)(p-c)}, p-a=\frac{-a+b+c}{2}$.

Arătăm că $S_{ABC}^2\leq\frac{1}{4a^2}$

$S_{ABC}^2=\frac{1}{16}\left(a+b+c\right)\left(-a+b+c\right)\left(a-b+c\right)\left(a+b-c\right)=\left(1p\right)$

$\frac{1}{16}\left((b+c)^2-a^2\right)\left(a^2-(b-c)^2\right)\left(1p\right)=\frac{1}{16}\left(b^2+c^2-a^2+2bc\right)\left(a^2-b^2-c^2+2bc\right)$

$=\frac{1}{16}\left(b^2+c^2-a^2+\frac{2}{a}\right)\left(a^2-b^2-c^2+\frac{2}{a}\right)=\frac{1}{16a^2}\left(2+ab^2+ac^2-a^3\right)\left(2+a^3-ab^2-ac^2\right)=$

$\frac{1}{16a^2}\left[4-(a^3-ab^2-ac^2)^2\right]\left(1p\right)\leq\frac{1}{16a^2}4=\frac{1}{4a^2}\left(1p\right)$. Avem $S_{ABC}\leq\frac{1}{2a}$, pentru

$\min\left\{\frac{1}{2a},\frac{1}{2b},\frac{1}{2c}\right\}=\frac{1}{2a}$ adevărat. $\left(1p\right)$